IOP Publishing European Journal of Physics
Eur. J. Phys. **28** (2007) 961–975 doi:10.1088/0143-0807/28/5/019

Oscillations of a string with concentrated masses

B J Gómez[1,2], C E Repetto[1,2], C R Stia[1,2] and R Welti[2]

[1] Instituto de Física Rosario (CONICET-UNR), Bv. 27 de Febrero 210 Bis, 2000 Rosario, Argentina
[2] Dto. de Física, Esc. de Formación Básica, Facultad de Ciencias Exactas, Ingeniería y Agrimensura, Universidad Nacional de Rosario, Av. Pellegrini 250, 2000 Rosario, Argentina

E-mail: weltireinaldo@arnet.com.ar

Received 2 March 2007, in final form 29 June 2007
Published 30 July 2007
Online at stacks.iop.org/EJP/28/961

Abstract
In this work, the oscillations of a homogeneous string fixed at both ends, and loaded with a finite number of masses, are studied. Through a simple device, the cases with one and two concentrated masses are analysed in detail. The normal modes are observed and the corresponding frequencies are recorded. The experimental results and the solutions of the wave equation that satisfy suitable boundary conditions are compared. The theoretical and experimental results are in very good agreement.

1. Introduction

Vibrating strings have been widely investigated during the 18th century by Euler, d'Alembert and Lagrange. Several problems in this domain were more difficult to solve than it might seem today, now that vibrations have become a topic in textbooks. For instance, there was a strong debate on the possibility of a slope discontinuity for a plucked string [1]. These different points of view can be considered as one of the first written evidences on the difficulties to be surmounted to arrive at the mathematical definition of 'function', a very familiar concept nowadays in science. Lagrange also examined the non-harmonic vibrations of a string with a variable density [1]. Helmholtz used a string loaded at the middle point to determine the exact frequency at which the impulses fuse into a continuous tone [2]. Rayleigh's calculations give the eigenfunctions and the eigenfrequencies of the system as a function of the mass position along the string [3].

The physics of the waves plays an important role in the education of physicists and engineers. Particularly, waves in strings are visible waves that provide a simple introduction to wave phenomena and can be helpful to understand more complex subjects. For that reason, vibrations of tense cords and, in particular, musical string instruments, have been extensively studied in many textbooks of university level. Nevertheless, even if specific solutions of the d'Alembert's equation are routine exercises for students, the Rayleigh's equations for the loaded string were rarely studied.

More recently, Chen has obtained the eigenfrequencies of a loaded string with a concentrated mass using the Dirac delta function and the Laplace-transform methods [4]. Pong has shown how the masses of the springs will affect the principal frequencies of a double spring–mass system [5]. Parmley *et al* [6] have studied by means of numerical calculations and experimentally the eigenfunctions and the corresponding eigenfrequencies of a chain mass string. Santos *et al* [7] have discussed the motion of a finite mass spring coupled to two pointlike masses fixed at its ends.

In this work, we study the normal modes of the oscillations of a homogeneous string, fixed at both ends, and loaded with one or two masses. In both cases, we consider that the masses are placed at symmetric positions on the string. We describe a simple experimental device that allows the observation of the form of the modes and to measure their frequencies. Bolwell analysed the free vibrations of a loaded string [8], but the physics on which the description is based is incorrect [9]. For that reason, in this work, we study in detail the free vibrations of a loaded string and we compare them with the free vibrations of a homogeneous string. The mathematical approach as well as the experimental device can be adapted easily to be employed in introductory and/or advanced courses of physics.

2. Oscillations of a loaded string

We study the problem of the transverse oscillations of a fixed string of length l, under tension T, in which N concentrated masses M_i have been placed at the points $x = x_i$ $(i = 1, \ldots, N)$. Such a problem can be considered as the equivalent one corresponding to a nonhomogeneous string with a linear mass density $\mu(x)$ given by

$$\mu(x) = \mu_0 + \sum_{i=1}^{N} M_i \, \delta(x - x_i) \tag{1}$$

where μ_0 is the constant mass density of the different string portions linking the point masses and δ is the Dirac's delta function.

The equation that describes the transverse oscillations of the nonhomogeneous string is

$$\frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) = \mu(x)\frac{\partial^2 u}{\partial t^2}. \tag{2}$$

The deflection function $u = u(x, t)$ represents the vertical displacement of the string at a distance x, measured from the point $x = x_0$, and at time t. Integrating equation (2) over the x coordinate, from $x_i - \epsilon$ to $x_i + \epsilon$, with $\epsilon > 0$, and taking then the limit as $\epsilon \to 0$, we obtain the following condition:

$$\left(T\frac{\partial u}{\partial x}\right)_{x_i-0}^{x_i+0} = M_i\frac{\partial^2 u(x_i, t)}{\partial t^2} \tag{3}$$

that represents a jump of the first derivative of the function u with respect to x, evaluated at $x = x_i$.

The other condition to be satisfied at the positions $x = x_i$ is the continuity of the displacements, namely,

$$u(x_i - 0, t) = u(x_i + 0, t). \tag{4}$$

The problem can be now formulated in the following form: find the solutions of the wave equation corresponding to a homogeneous string,

$$\frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) = \mu_0 \frac{\partial^2 u}{\partial t^2}, \tag{5}$$

satisfying both the boundary conditions,

$$u(x_0, t) = 0 \qquad u(x_{N+1}, t) = 0, \tag{6}$$

and the additional conditions (3) and (4), where $x_{N+1} = x_0 + l$ is the end of the string.

Using the method of separation of variables, we consider the solutions of equation (5) under the form

$$u(x, t) = \Phi(x)\,\mathrm{e}^{\mathrm{i}\omega t} \tag{7}$$

where ω is the angular frequency.

Replacing the last expression into equation (5), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(T\frac{\partial \Phi}{\partial x}\right) + \omega^2 \mu_0 \Phi = 0. \tag{8}$$

Similarly, replacing equation (7) into equations (6), (4) and (3), the following conditions are obtained:

$$\Phi(x_0) = 0 \qquad \Phi(x_{N+1}) = 0 \tag{9}$$

$$\Phi(x_i - 0) = \Phi(x_i + 0); \qquad \left(T\frac{\partial \Phi}{\partial x}\right)_{x_i-0}^{x_i+0} = -M_i\omega^2\Phi(x_i) \tag{10}$$

for $i = 1, 2, \ldots, N$.

Equation (8) and the conditions given by equations (9) and (10) constitute a problem of eigenvalues. It is worth mentioning that the parameter ω appears not only in equation (8), but also in the complementary conditions (10).

The eigenfunctions Φ_n of the boundary problem,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(T\frac{\partial \Phi}{\partial x}\right) + \omega^2 \mu(x) \Phi = 0; \qquad \Phi(0) = 0, \quad \Phi(l) = 0, \tag{11}$$

are orthogonal, with density μ, on the interval $(0, l)$:

$$\int_0^l \Phi_m(x)\Phi_n(x)\mu_0\,\mathrm{d}x + \sum_{i=1}^{N} M_i\Phi_m(x_i)\Phi_n(x_i) = 0 \tag{12}$$

being $m \neq n$ [10, 11]. Finally, a general solution $f(x)$ can be expanded in terms of the eigenfunctions Φ_n:

$$f(x) = \sum_{i=1}^{\infty} A_n\Phi_n(x) \tag{13}$$

where the coefficients A_n are determined by the equation

$$A_n = \frac{\int_0^l f(x)\Phi_n(x)\mu_0 dx + \sum_{i=1}^{N} M_i f(x_i)\Phi_n(x_i)}{\|\Phi_n\|^2} \tag{14}$$

with $\|\Phi_n\|$, the norm of the eigenfunctions Φ_n [10, 11].

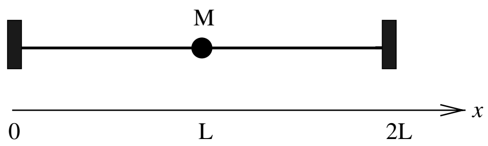

Figure 1. Single-mass-loaded string of length $2L$ and linear mass density μ_0. The mass M is attached at the point $x = L$.

2.1. A single-mass-loaded string

First, we consider the simplest case of a homogeneous string of length $l = 2L$ with linear mass density μ_0 and both fixed ends at $x = 0$ and $x = 2L$. We also consider that the string is loaded with a single point mass M at $x = L$, as shown in figure 1. Moreover, it is supposed that the tension T of the string is uniform at the rest position. Outside this position, as the string oscillates, its length increases and, as a consequence, the value of the tension changes. However, if $\partial\Phi/\partial x \ll 1$, the increment of the string length is small and it can be supposed that the tension remains constant, assuming the same value T as at the rest position [12, 13]. In this case, the solution of equation (8), satisfying the proper boundary conditions at the points $x = 0$ and $x = 2L$ (see equation (9)), can be represented as follows:

$$\Phi(x) = \begin{cases} A\sin(kx) & 0 \leqslant x \leqslant L \\ B\sin(k(x-2L)) & L \leqslant x \leqslant 2L \end{cases} \tag{15}$$

where $k = \omega/c$, and $c = \sqrt{T/\mu_0}$ is the velocity of the travelling waves.

Conditions (10) are satisfied whenever

$$F_1(kL)F_2(kL) = 0 \tag{16}$$

where

$$F_1(kL) = \sin(kL) \qquad F_2(kL) = p\sin(kL) - 2\cos(kL) \tag{17}$$

and $p = M\omega^2/Tk = (M/m)kL$, and $m = \mu_0 L$ is the mass of each one of the string portions.

Equation (16) is the product of two factors which give place to two transcendental equations. Consequently, two different eigenvalue sets will be obtained [4]. As we will show below, the eigenvalues (and eigenfunctions) related to each one of these factors are physical and mathematically very different.

The eigenvalues related to F_1 are easily found to be

$$k_{n'}L = n'\pi \tag{18}$$

where n' is a natural number. For this eigenvalue set it is verified that $B = A$. Replacing this result in equation (15) and also considering equation (18), the corresponding eigenfunctions are given by

$$\Phi_{n'}(x) = A\sin(k_{n'}x); \qquad 0 \leqslant x \leqslant 2L. \tag{19}$$

The eigenfunctions (19) coincide with the normal vibration modes of a homogeneous string of length $2L$, having an even index. Such modes present a node at $x = L$ and, consequently, they are not altered by the presence of the concentrated mass at this position. These modes with a continuous first derivative of the displacements at $x = L$ are symmetric with respect to $x = L$. Hereinafter, we will briefly refer to these modes as eigenmodes of *continuous slope*.

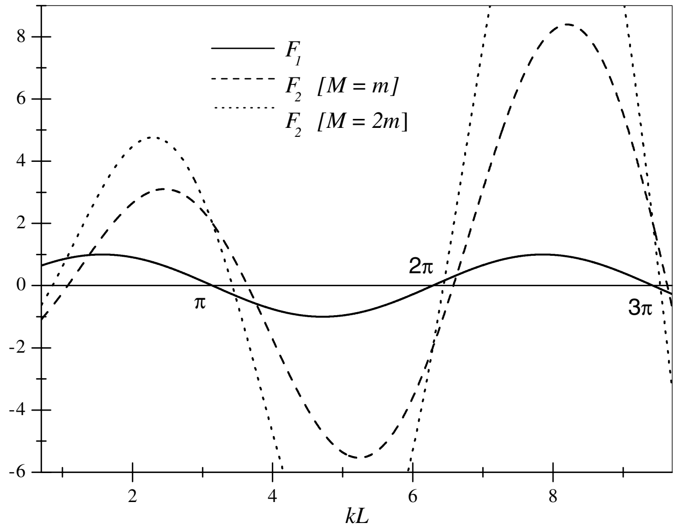

Figure 2. Factors F_1 and F_2 for different values of mass M.

Table 1. First theoretical eigenvalues related to a vibrating homogeneous string, whose length is $2L$, with a mass M attached at the point $x = L$. These values were obtained as roots of the F_1 and F_2 factors (see the text). The linear mass density of the string is $\mu_0 = m/L$.

	k_n		
n	F_1	$F_2\ (M = 2m)$	$F_2\ (M = m)$
1		0.860	1.067
2	π		
3		3.425	3.643
4	2π		
5		6.437	6.578
6	3π		
7		9.529	9.629

To determine the eigenvalues related to the factor F_2 in equation (16), a graphical method is chosen here. In figure 2, we represent the factor F_2 as a function of kL. Thus, the eigenvalues are determined as zeros of this function. In this case, it is found that the relation $B = -A$ between wave amplitudes is satisfied. The related eigenfunctions now present a jump of the first derivative of the transverse displacement at $x = L$, and we will refer to them as eigenmodes of *discontinuous slope*. These modes are antisymmetric with respect to $x = L$.

To sum up, we denote with Φ_n all the eigenfunctions of the boundary problem, with k_n being the corresponding eigenvalues. In table 1, we show the eigenvalues corresponding to the first modes. The modes of odd index n are *discontinuous-slope* modes, while those of even index n are modes of *continuous slope*. As can be seen from table 1, both kinds of modes successively alternate, being a *discontinuous-slope* mode that with the lowest frequency.

The ratio between the frequencies f_2 and f_1 corresponding to the second and first modes, respectively, depends strongly on the ratio between masses M/m. For instance, for the case $M/m = 1$, $f_2/f_1 = 2.917$ is found, while for the case $M/m = 2$, the ratio f_2/f_1 increases, and the obtained value is 3.653. This fact has been used by Helmholtz [2] in the studies on

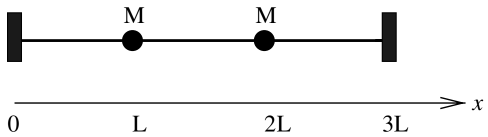

Figure 3. Homogeneous string of length $3L$ and linear mass density μ_0, loaded with two masses M at $x = L$ and $x = 2L$.

the sensations of the tone and with the objective of separating the first two harmonics. From figure 2, it can be seen that both sets of the eigenvalues, i.e., those obtained as roots of the F_1 and F_2 factors in equation (16), become nearly identical as n increases enough.

For the case $M \gg m$, it is found from equation (17) that only the even modes of a homogeneous string of $2L$ length are obtained as solutions of the problem of the loaded string. In contrast, for the case $M \ll m$, all the modes of a homogeneous vibrating string are found.

2.2. Two-mass-loaded string

We now consider the case corresponding to a $3L$ length homogeneous string with linear mass density μ_0 and both fixed ends ($x = 0$ and $x = 3L$), that is loaded with two point masses M at $x = L$ and $x = 2L$, as sketched in figure 3. The tension T is also considered constant here as in the precedent case.

The solution of the differential equation (8), satisfying the boundary conditions at $x = 0$ and $x = 3L$ (equation (9)), can be represented as follows:

$$\Phi(x) = \begin{cases} A\sin(kx) & 0 \leqslant x \leqslant L \\ B\sin(k(x-L)) + C\cos(k(x-L)) & L \leqslant x \leqslant 2L \\ D\sin(kL) & 2L \leqslant x \leqslant 3L. \end{cases} \tag{20}$$

Conditions (10) to be satisfied at $x = L$ and $x = 2L$ lead to

$$F_1(kL)F_2^{'}(kL)F_3^{'}(kL) = 0 \tag{21}$$

where

$$\begin{aligned} F_1 &= \sin(kL) \\ F_2^{'} &= p\sin(kL) - 2\cos(kL) - 1 \\ F_3^{'} &= p\sin(kL) - 2\cos(kL) + 1. \end{aligned} \tag{22}$$

The eigenvalues related to the factor F_1 in equation (21) are

$$k_{n'}L = n'\pi \tag{23}$$

where n' is a natural number. Replacing equation (23) into equation (10), it follows that

$$B = (-1)^{n'}A, \qquad C = 0, \qquad D = (-1)^{n'}A. \tag{24}$$

Replacing this result in (20), we obtain that

$$\Phi_{n'}(x) = A\sin(k_{n'}x) \qquad 0 \leqslant x \leqslant 3L. \tag{25}$$

The eigenvalues (23) and the eigenfunctions (25) coincide with the natural modes and frequencies multiple of 3 ($3n$, $n = 1, 2, 3, \ldots$) corresponding to a uniform string of $3L$ length. As these modes have nodes at $x = L$ and $x = 2L$, they are not modified by the presence of the masses at these points. In the same way as for the case of a single mass, these modes

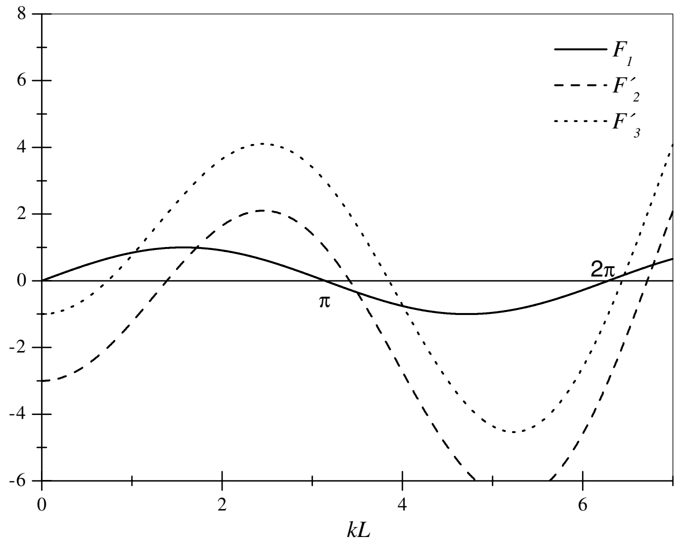

Figure 4. Plots of the factors F_1, F_2' and F_3' for $M = m$.

will be called modes of *continuous slope* because they have continuous first derivative of the displacements at the points $x = L$ and $x = 2L$.

The eigenvalues related to the second and third factors in equation (21) are determined directly from figure 4. In figure 4, F_2' and F_3' are represented as functions of kL. The function $F_1(kL)$ is also included in the figure. On the one hand, for the eigenvalues related to F_2', it is found that $B = -(1 - \cos(kL))A$, $C = A\sin(kL)$ and $D = A$, and the corresponding eigenfunctions result antisymmetric with respect to the middle point $x = 3L/2$:

$$\Phi(x) = \begin{cases} A\sin(kx) & 0 \leqslant x \leqslant L \\ 2A\cos(kL/2)\sin(k(3L/2 - x)) & L \leqslant x \leqslant 2L \\ -A\sin(k(3L - x)) & 2L \leqslant x \leqslant 3L. \end{cases} \tag{26}$$

On the other hand, the eigenvalues associated with the factor F_3' lead to $B = (1 + \cos(kL))A$, $C = A\sin(kL)$ and $D = -A$, and the related eigenfunctions are symmetric with respect to the middle point of the string. In this case,

$$\Phi(x) = \begin{cases} A\sin(kx) & 0 \leqslant x \leqslant L \\ 2A\sin(kL/2)\cos(k(3L/2 - x)) & L \leqslant x \leqslant 2L \\ A\sin(k(3L - x)) & 2L \leqslant x \leqslant 3L. \end{cases} \tag{27}$$

However, both eigenfunction sets (equations 26 and 27) correspond to modes of *discontinuous slope* at the points where the masses were loaded, i.e., $x = L$ and $x = 2L$.

3. Experimental set-up

The experimental device was mounted in order to study the forced vibrations of a string with both fixed ends and loaded with one or two masses as shown in figure 5. This enables us to identify the normal modes describing the transverse string vibration and then to compare them with those expected from the theory.

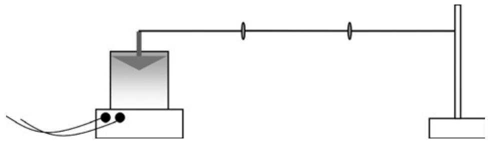

Figure 5. Experimental device used.

We used washers of mass M as concentrated masses that were joined to identical springs of mass m (we chose $M = m$). For the case of a single-concentrated-mass set-up, $M = 6.9\ g$ was used, whereas for the case of two masses, $M = 4.2\ g$ was employed. Springs instead of rubber elastic strings were used in order to enhance the amplitude of the transverse oscillations. Strings were found to possess a high friction coefficient that produces strong damping oscillations.

We worked with a Pasco Scientific WA-9753 mechanical wave driver. The wave driver was connected to a PC through an interface PASCO 750, and using the software Science Workshop it was possible to handle a generator of functions and to change the values of amplitude and frequency of the different signals. So, in both cases studied here, one of the ends of the springs was attached to a support and the other one was connected to the piston rod of the wave driver. In this way, the system springs–masses and the wave driver were vibrating at the same frequency. The length l of the whole system was particularly chosen in such a way that the vibrational modes were clearly observed.

A tension of 3 V was applied to the driver, implying maximum displacements of the piston rod of 3 mm approximately. These displacements were much smaller than the maximum amplitudes of the string oscillations. In particular, when the frequency of the wave driver is equal to one of the normal frequencies of the system, the corresponding normal mode of oscillation is excited. In this case, the amplitudes of the transverse oscillations are greater than the displacements of the piston road. For that reason, we can also consider this point as a fixed end [13].

Finally, we varied the frequencies of the external excitation in order to obtain the different vibrational modes for both configurations that we have considered. It is important to emphasize that we have considered as the frequency of each normal mode the value of the external frequency for which the amplitude of oscillations is maximum (resonance). The resonance peaks were found to be very sharp, showing a high sensitivity of the experimental device ($\approx$0.01 Hz).

4. Results

In the following, we present our results corresponding to the resonant modes of the string fixed at its two ends and loaded with one or two masses M. In the resonant condition, the frequency of the external excitation matches one of the normal frequencies of the system, exciting the corresponding normal vibration mode. Several resonances corresponding to the different vibration modes are then expected here. In this way, we are able to explore the different vibration modes and to compare them with the predictions of the theory shown in section 2.

4.1. One concentrated mass

We first analyse the resonant modes of the system with a single mass M attached at its middle point. In table 2, we show the measured external frequencies corresponding to the seven first resonant modes of the system. We have also included in table 2 the theoretical ratios of the

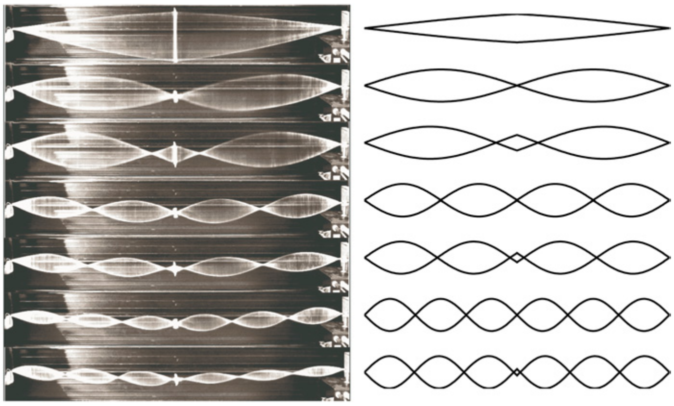

Figure 6. Normal modes of a homogeneous string with a concentrated mass in the middle point. Left panel: experimental results. Right panel: analytical solutions.

Table 2. Measured normal frequencies for the system with one concentrated mass. The theoretical ratios are constructed from the normal frequencies shown in table 1, for the case $M = m$.

Frequencies (Hz)	Experimental ratio	Theoretical ratio
$f_1 = 6.1$		
$f_2 = 17.7$	$f_2/f_1 = 2.902$	$f_2/f_1 = 2.917$
$f_3 = 21.0$	$f_3/f_1 = 3.443$	$f_3/f_1 = 3.383$
$f_4 = 35.5$	$f_4/f_1 = 5.820$	$f_4/f_1 = 5.835$
$f_5 = 38.0$	$f_5/f_1 = 6.229$	$f_5/f_1 = 6.109$
$f_6 = 53.2$	$f_6/f_1 = 8.721$	$f_6/f_1 = 8.752$
$f_7 = 55.71$	$f_7/f_1 = 9.133$	$f_7/f_1 = 8.942$

different normal frequencies and the fundamental one. These frequencies correspond to those obtained from equations (16) and (17) (see table 1, the case $M = m$). A very good agreement is found between theory and experiments. In figure 6, the pictures of the observed modes are shown. The theoretical eigenfunctions computed from equation (15) are also included in figure 6. As expected, the modes of continuous and discontinuous slopes alternate, the fundamental mode being a discontinuous slope.

As can be noticed from table 2, there are consecutive modes having close frequencies. What happens if we excite the system with an intermediate external frequency, lying between two of these consecutive frequencies? It is well known from the forced oscillations that, after a transitory regime, the system oscillates with the frequency of the external excitation. But the motion is not trivial, since these consecutive modes are excited in a different manner: the string motion corresponding to the mode with smaller frequency is in phase opposition with respect to the movement of the piston rod, whereas that of greater frequency is in phase with the movement of the piston rod. In addition, as it was mentioned previously, the eigenfunctions that describe consecutive modes are either symmetric or antisymmetric functions with respect to the centre of the system. Then, if we excite the system with a concentrated mass at a frequency of 19.3 Hz, the second and third modes will be simultaneously excited. According to the previous discussion, half of the system will be practically horizontal since the contributions of the two modes are cancelled, whereas in the other half those contributions are reinforced, as can be seen in figure 7. Similar behaviour may be obtained if the

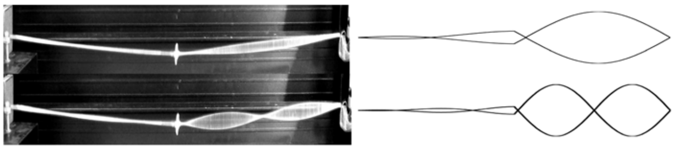

Figure 7. Oscillations of a homogeneous string with a concentrated mass in the middle point. The system is excited with an intermediate external frequency, lying between: the second and third modes (top); and between the fourth and fifth modes (bottom). Left panel: experimental results. Right panel: theoretical solutions.

Table 3. Frequencies measured for the system with two concentrated masses. The theoretical ratios are constructed from the frequencies obtained from equation (21).

Frequencies (Hz)	Experimental ratio	Theoretical ratio
$f_1 = 3.76$		
$f_2 = 7.09$	$f_2/f_1 = 1.886$	$f_2/f_1 = 1.897$
$f_3 = 15.80$	$f_3/f_1 = 4.202$	$f_3/f_1 = 4.295$
$f_4 = 17.52$	$f_4/f_1 = 4.660$	$f_4/f_1 = 4.670$
$f_5 = 19.98$	$f_5/f_1 = 5.316$	$f_5/f_1 = 5.268$
$f_6 = 31.60$	$f_6/f_1 = 8.404$	$f_6/f_1 = 8.590$
$f_7 = 33.11$	$f_7/f_1 = 8.806$	$f_7/f_1 = 8.799$
$f_8 = 34.80$	$f_8/f_1 = 9.255$	$f_8/f_1 = 9.182$
$f_9 = 46.73$	$f_9/f_1 = 12.428$	$f_9/f_1 = 12.886$

system is excited at a frequency of 36.4 Hz but in this case the fourth and fifth modes are excited simultaneously.

4.2. Two concentrated masses

In table 3, we show the measured frequencies for the system with two concentrated masses. Analogously with the previous case, in table 3 we have also included both the experimental and theoretical ratios of the different frequencies and the fundamental one, to make possible a comparison between theory and experience. These theoretical frequencies correspond to those obtained from equation (21).

In figure 8, the pictures of the nine first normal modes of a fixed string loaded with two point masses are shown. Theoretical eigenfunctions computed from equation (20) are also included in the figure to compare with the experiments. In this case, it is observed that the discontinuous and continuous slope modes alternate in the following way: every two modes of discontinuous slope are followed by one with a continuous slope.

As can be noted from table 3, there exist consecutive modes with nearby frequencies. If the system is excited with an intermediate external frequency, the general motion of the string may be described in a similar way as the case of one concentrated mass.

In the case of massless strings, the system will have only two normal modes of oscillations. These modes correspond to both masses oscillating in phase or in phase opposition. It is easy to show that the ratio of the corresponding frequencies is equal to $\sqrt{3}$. The form of these modes corresponds to the two first modes shown in figure 8. In the present case, in which the mass m of the string is non negligible, the ratio $f_2/f_1 \approx 1.9$ (see table 3). However, the ratio f_2/f_1 tends to $\sqrt{3}$ as M increases.

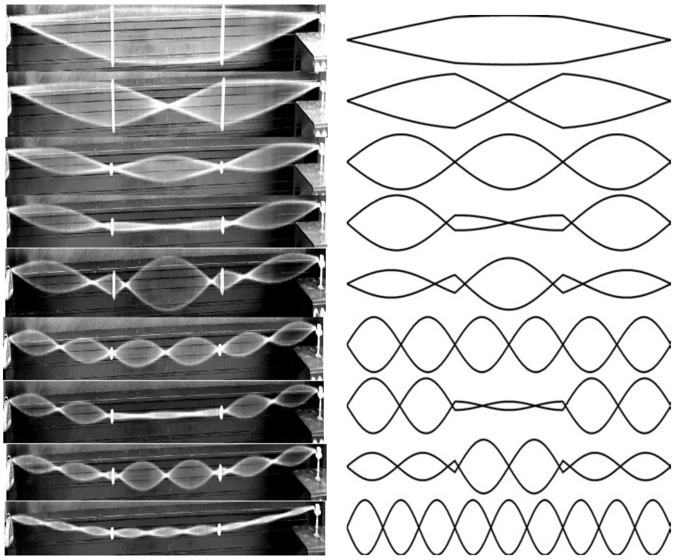

Figure 8. Normal modes of a homogeneous string with two concentrated masses. (a) Experimental results. (b) Analytical solutions.

5. Free oscillations

We now study the free oscillations of a fixed string of density $\mu(x)$ and length $2L$, with a mass M attached at $x = L$. In this case, the eigenfunctions Φ_n, solutions of the corresponding boundary problem, are given by equation (15), while the corresponding eigenvalues are determined from the transcendental equation (16).

The general solution of the problem can be written as a linear combination of the eigenfunctions Φ_n:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \Phi_n(x) \cos(ck_n t + \phi_n). \tag{28}$$

Equation (28) represents the general motion of a fixed string loaded at its centre. The phases ϕ_n and the amplitudes A_n in equation (28) are determined from the initial conditions:

$$u(x,0) = f(x) \tag{29}$$

$$\dot{u}(x,0) = g(x) \tag{30}$$

where both $f(x)$ and $g(x)$ are arbitrary well-behaved functions, describing the position and the velocity of the string at $t = 0$, respectively. Supposing that the initial velocity of the string is equal to zero, i.e., $g(x) = 0$, it follows that $\phi_n = 0$ for all natural n. Taking into account this result, condition (29) then gives

$$f(x) = \sum_{n=1}^{\infty} A_n \Phi_n(x). \tag{31}$$

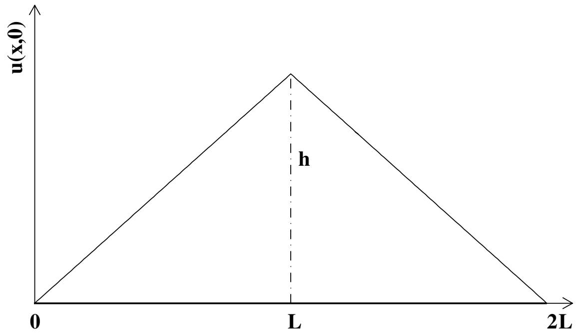

Figure 9. Initial shape of a uniform string fixed at both ends.

Equation (31) represents the expansion of the function $f(x)$ in terms of the eigenfunctions $\Phi_n(x)$. The coefficients A_n can be determined by using equation (14). In the following, we consider the case where the mass M is initially separated a distance h from the rest position, as shown in figure 9. In this case, the function $f(x)$ takes the simple form

$$f(x) = \begin{cases} \dfrac{h}{L}x & 0 \leqslant x \leqslant L \\ -\dfrac{h}{L}x + 2h & L \leqslant x \leqslant 2L. \end{cases} \tag{32}$$

Using equation (14), the coefficients A_n are found to be

$$A_n = \begin{cases} \dfrac{2h\sin(k_nL)}{A(k_nL)^2\left[1 + \frac{M}{2m}\sin^2(k_nL)\right]} & n = 1, 3, 5\ldots \\ 0 & n = 2, 4, 6\ldots. \end{cases} \tag{33}$$

The result (33) indicates that only the odd-index modes contribute to the general motion of a string pulsed at its centre. This result can be understood observing that the initial shape of the string $f(x)$ is a symmetric function with respect to the point $x = L$. The normal modes associated with odd index appearing in (33) are *discontinuous-slope* modes and, as a consequence, they are also symmetric functions with respect to $x = L$.

The velocity of an arbitrary point x_0 of the string, as a function of time, is defined as follows:

$$v_y(x_0, t) = \frac{\partial u(x_0, t)}{\partial t}. \tag{34}$$

In particular, choosing $x_0 = L$ and taking into account equations (28)–(34) and $\Phi_n(L) = A\sin(k_nL)$, we obtain

$$v_y(L, t) = -\frac{2hc}{L^2}\sum_{i=1}^{\infty}\frac{\sin^2(k_nL)}{k_n\left[1 + \frac{M}{2m}\sin^2(k_nL)\right]}\sin(ck_nL) \tag{35}$$

with n an odd natural number.

The velocity of the $x = L$ point for the case of a uniform fixed string can be easily derived from equation (35) by taking $M = 0$. In this case, $k_nL = n\pi/2$ is satisfied for all natural n. Then,

$$v_y^{(h)}(L, t) = -\frac{4hc}{\pi L}\sum_{i=1}^{\infty}\frac{\sin(nck_1t)}{n} \tag{36}$$

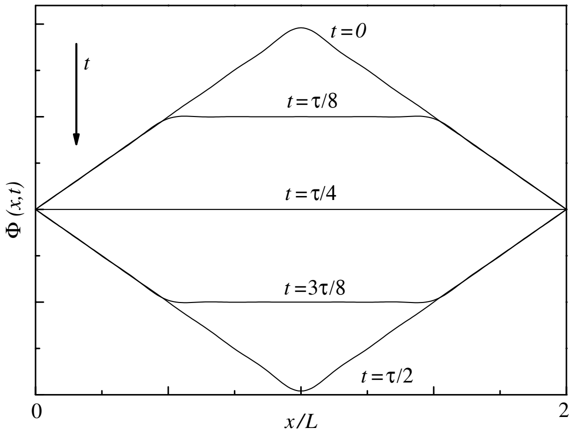

Figure 10. Snapshots of the free vibrations of a uniform string displaced at its centre and released from initial velocity zero. After a time $t = \tau$ the initial shape of the string is recovered.

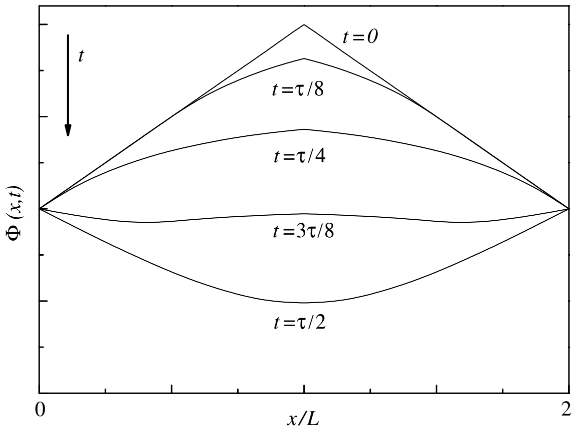

Figure 11. Idem figure 10 but for the case of a uniform string with a mass $M = m$ attached at its centre, with m being the mass of each string portion. τ is the period of the vibrations of a uniform string without attached mass.

where the upper index (h) indicates that it corresponds to the case of a homogeneous string, of $2L$ length, without an attached mass. Equation (36) represents the Fourier's expansion of a periodic square wave of height $\pm 4ch/\pi L$.

In figure 10, different snapshots at $t = 0$, $\tau/8$, $\tau/4$, $3\tau/8$ and $\tau/2$ of a homogeneous string without concentrated mass are shown, where $\tau = 4L/c$ is the period of oscillation of the string of length $2L$. In figure 11, different snapshots at the same times for the string with an $M = m$ concentrated mass on its middle point are shown. For the calculation of the free oscillations of the homogeneous string the first ten odd harmonics of the series were considered, whereas for the case of the concentrated mass, the first seven terms were used in formula (28). The inclusion of higher order harmonics does not modify the results discussed here.

In figure 10, it is observed that the points of the string initiate the motion with a constant velocity, which coincides with the velocity of the middle point given by equation (36). This

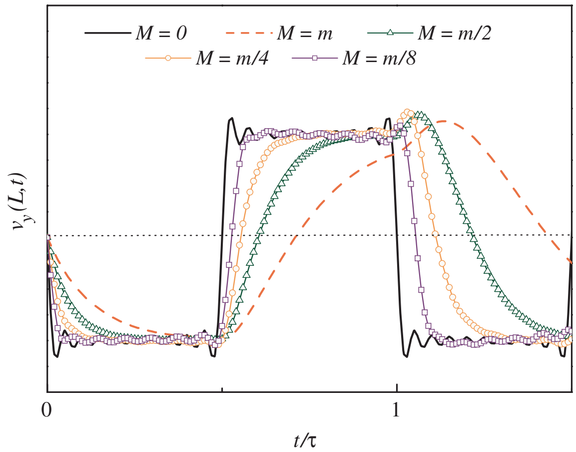

Figure 12. Transverse velocity at $x = L$ of the string loaded with a single mass M at that point, as a function of the time. m is the mass of each string portion of length L.

(This figure is in colour only in the electronic version)

behaviour can be explained as it follows. In the case of the homogeneous string, an element of a string of infinitesimal mass located in its middle point experiments at $t = 0$ a transverse net force pointing down. In this moment this string element acquires an instantaneous infinite acceleration that produces a jump in the velocity from 0 to $-(4h/\rho L)c$. Immediately after, the net force acting on this element of string turns to zero and, therefore, it moves with a uniform velocity $-(4h/\rho L)c$ until it reaches $t = \tau/2$ at the lowest position. Then, a net force pointing upwards acts producing a reverse of the velocity without changing the module. The transverse velocity of the middle point of the string as a function of time is a square wave of period τ.

In the case of the loaded string the behaviour is different. Due to the presence of the attached mass, the movement of the system is not periodic: the initial form is not retrieved anymore. Moreover, the motion is delayed with respect to the one corresponding to a homogeneous string (see figure 11). This delay increases as the value of the attached mass increases.

In figure 12, the transverse velocity of the point L is represented for different values of the concentrated mass. As the acceleration of the concentrated mass is always finite, its velocity has no discontinuities. The case $M = 0$ that corresponds to the homogeneous string is also included. It can be observed that the transverse velocity for the case of the loaded string tends to that of the homogeneous string as M diminishes, recovering the square wave that characterizes the transverse velocity for $M = 0$. The oscillations observed for the case $M = 0$ in figure 12 are simply due to the inclusion of a small number of harmonics.

6. Conclusion

In this work, we have analysed the normal modes of oscillation of a loaded string with one or two masses and fixed at its ends. We found the correct eigenfunctions of the problem by using the correct orthogonality conditions, at variance with those given in [8].

The experimental set-up allowed us to clearly visualize the different vibrational modes. The eigenvalues calculated analytically and the corresponding eigenfunctions were compared with those obtained experimentally, showing a very good agreement. It has also been possible to observe the particular shape that acquires the system when it is excited with a frequency, lying between two consecutive modes, which also shows a good agreement with the analytic solution.

We have compared in detail the free vibrations of a loaded string pulsed at its centre with the case of the free vibrations of a homogeneous string. The transverse velocity of the middle point of the system as a function of time was analysed for different values of the concentrated mass, including the case $M = 0$.

Finally, it is noted that the problem studied here may be useful to gain further insight into the comprehension of other different physical systems. For instance, it may be shown that the eigenvalue problem related to the ammonia molecule [14] has close similarities with the string loaded at its middle point. It is also the case of a dipole antenna with concentrated inductances. Concentrated inductances can be used in order to reduce the physical length of the antenna in its low frequency mode [15]. It can be shown that the effect of the concentrated inductance is similar to the concentrated mass in a homogeneous string analysed in this work. These subjects will be included in a future article.

Acknowledgments

The authors would like to thank the Facultad de Ciencias Exactas, Ingeniería y Agrimensura and the Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina, for financial support.

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